ORBIT METHOD FOR NILPOTENT LIE GROUPS

ABHINAV CHAND

ABSTRACT. This paper presents an overview of some parts of the orbit method for Nilpotent Lie groups as outlined by Kirillov. We begin by proving the existence of symplectic structure on coadjoint orbits. Then we proceed to sketch I.D. Brown's proof that the unitary dual and coadjoint orbit space for connected, simply connected nilpotent Lie groups homeomorphic. Finally, we illustrate Kirillov's correspondence for the Heisenberg group.

CONTENTS

1. Introduction 1
2. Coadjoint Orbits 2
3. Canonical form σ_Ω 3
4. Symplectic structure on coadjoint orbits 4
5. Invariant polarization on homogeneous symplectic manifolds 5
6. Orbit Method for Nilpotent Lie Groups 6
7. Example of Heisenberg Group 9
References 11

1. INTRODUCTION

The orbit method was first proposed by Kirillov for the description of the unitary dual of nilpotent Lie groups. It turned out that the method not only solves the problem of finding the unitary dual but also gives simple and visual solutions to all other principal questions in representation theory: topological structure of the unitary dual, explicit description of the restriction and induction functors, the formulae for generalized and infinitesimal characters, the computation of the Plancherel measure, etc.

The reader can refer to the "User's Guide" outlined in Pg.xix of Kirillov's book [1]. Here I will present only two parts of that user's guide, and present some of it's aspects for nilpotent Lie groups.

What you want	What you have to do
Describe the unitary dual $\hat{G}$ as a topological space	Take the space $\mathcal{O}(G)$ of coadjoint orbits with the quotient topology.
Construct the unirrep π_Ω associated to the orbit $\Omega \in \mathfrak{g}^*$.	Choose a point $F \in \Omega$, take a subalgebra $\mathfrak{h}$ of maximal dimension subordinate to F, and put $\pi_\Omega = Ind_H^G \rho_{F,H}$.

2. Coadjoint Orbits

We will start with the definitions for our setup. Let G be a Lie group. Let $\mathfrak{g}$ denote the Lie algebra associated to G. The group G acts on itself by inner automorphism: $A(g) : x \mapsto gxg^{-1}$. The identity e is a fixed point of this action and thus we have the induced map: $(A(g))_*(e) : \mathfrak{g} \to \mathfrak{g}$. This map is denoted by $Ad(g)$. The map $g \mapsto Ad(g)$ is called the adjoint representation of G.

Consider the vector space $\mathfrak{g}^*$ dual to $\mathfrak{g}$. We obtain the dual of the adjoint representation as follows:

$$Ad^*(g) := Ad(g^{-1})^*$$

Here the asterisk in the right hand side means the dual operator in $\mathfrak{g}^*$. Thus we have the **coadjoint representation** $G \to GL(\mathfrak{g}^*)$ given by $g \mapsto Ad^*(g)$.

We use the special notation $K(g)$ to denote $Ad^*(g)$. So, by definition,

$$\langle K(g)F, X \rangle = \langle F, Ad(g^{-1})X \rangle$$

where $X \in \mathfrak{g}$, $F \in \mathfrak{g}^*$, and by $\langle F, X \rangle$ we denote the value of a linear function F on a vector X.

For matrix groups we can use the fact the the space $M_n(\mathbb{R})$ has a bilinear form

$$(A, B) = tr(AB)$$

which is non-degenerate and invariant under conjugation. Hence, the space $\mathfrak{g}^*$, dual to the subspace $\mathfrak{g} \subset M_n(\mathbb{R})$, can be identified with the quotient space $M_n(\mathbb{R})/\mathfrak{g}^\perp$. $\mathfrak{g}^\perp$ is the orthogonal complement of $\mathfrak{g}^\perp$ with respect to the bilinear form:

$$\mathfrak{g}^\perp = \{A \in M_n(\mathbb{R}) | (A, B) = 0 \text{ for all } B \in \mathfrak{g}\}$$

In practice the quotient space is often identified with a subspace $V \subset M_n(\mathbb{R})$ that is transversal to $\mathfrak{g}^\perp$ and has the complementary dimension. Therefore, we can write $M_n(\mathbb{R}) = V \oplus \mathfrak{g}^\perp$. Let p_V be the projection of $M_n(\mathbb{R})$ onto V parallel to $\mathfrak{g}^\perp$. Then the coadjoint representaiton K can be written in a simple form

$$K(g) : F \mapsto p_V(gFg^{-1}).$$

Example. *G= upper triangular matrices $g \in GL(n, \mathbb{R})$. Then $\mathfrak{g}$= upper triangular matrices in $M_n(\mathbb{R})$ and $\mathfrak{g}^\perp$= strictly upper triangular matrices. We can take for V the space of all lower triangular matrices. Hence, the coadjoint representation takes the form*

$$K(g) : \quad F \mapsto (gFg^{-1})_{lower\ part}$$

Example. *$G = SO(n, \mathbb{R})$. Then $\mathfrak{g}$ consists of all skew-symmetric matrices $X = -X^T$. Here we can put $V = \mathfrak{g}$ and omit the projection p_V. Hence,*

$$K(g)X = gXg^{-1}$$

Finally, we will give the formula for the corresonding representation K_* of the Lie algebra $\mathfrak{g}$ in $\mathfrak{g}^*$:

$$\langle K_*(X)F, Y\rangle = \langle F, -ad(X)Y\rangle = \langle F, [Y, X]\rangle$$

3. Canonical form σ_Ω

It is remarkable that all coadjoint orbits are symplectic manifolds. Moreover, each coadjoint orbit possesses a canonical G-invariant symplectic structure. We use the fact that a G-invariant differential form ω on a homogeneous G-manifold $M = G/H$ is uniquely determined by its value at the initial point m_0 and this value can be any H-invariant antisymmetric polylinear form on the tangent space $T_{m_0}M$.

Hence, to define σ_Ω it is enough to specify its value at some point $F \in \Omega$, which must be an antisymmetric bilinear form on $T_F\Omega$ invariant under the action of the group $StabF$.

Let $stab(F)$ be the Lie algebra of $Stab(F)$. We can consider the group G as a fiber bundle over the base $\Omega \cong G/Stab(F)$ with projection

$$p_F : G \to \Omega, \quad p_F(g) = K(g)F.$$

It is clear that the fiber above the point F is exactly $Stab(F)$. Consider the exact sequence of vector spaces

$$0 \to stab(F) \to \mathfrak{g} \xrightarrow{(p_F)_*} T_F(\Omega) \to 0$$

that comes from the above interpretation of G as a fiber bundle over Ω. It allows us to identify the tangent space $T_F(\Omega)$ with the quotient $\mathfrak{g}/stab(F)$.

Now we define the antisymmetric bilinear form B_F on $\mathfrak{g}$ by the formula

$$B_F(X, Y) = \langle F, [X, Y]\rangle$$

Lemma 3.1. *The kernel of B_F is exactly $stab(F)$.*

Proof.

$$\begin{aligned} kerB_F &= \{X \in \mathfrak{g} \mid B_F(X, y) = 0 \ \forall Y \in \mathfrak{g}\} \\ &= \{X \in \mathfrak{g} \mid \langle K_*(X)F, Y\rangle = 0 \ \forall Y \in \mathfrak{g}\} \\ &= \{X \in \mathfrak{g} \mid K_*(X)F = 0\} \\ &= stab(F) \end{aligned}$$

□

Lemma 3.2. *The form B_F is invariant under $Stab(F)$.*

Proof.

$$\langle F, [AdhX, AdhY]\rangle = \langle F, Adh[X, Y]\rangle = \langle K(h^{-1})F, [X, Y]\rangle = \langle F, [X, Y]\rangle$$

for all $h \in Stab(F)$ □

Definition 3.3. *Let Ω be a coadjoint orbit in $\mathfrak{g}^*$. We define the differential 2-form σ_Ω on Ω by*

$$\sigma_\Omega(F)(K_*(X)F, K_*(Y)F) = B_F(X, Y)$$

Note that Lemma 1.1 shows that this form is non-degenerate on $\Omega \cong \mathfrak{g}/stab(F)$, and Lemma 1.2 shows that the form is G invariant.

4. Symplectic structure on coadjoint orbits

In the previous section we defined the differential 2-form on the coadjoint orbits. Now we will show that it is closed, and hence a symplectic form.

Theorem 4.1. *The form σ_Ω is closed, hence defines on Ω a G-invariant symplectic structure.*

Proof. Using the Leibniz Rule for differential forms we can derive the following formula:

$$d\sigma(\xi, \eta, \zeta) = \circlearrowleft \xi\sigma(\eta, \zeta) - \circlearrowleft \sigma([\xi, \eta], \zeta)$$

where the sign $\circlearrowleft$ denotes the summation over cyclic permuations of ξ, η, ζ.

Let ξ, η, ζ be the vector fields on Ω which correspond to elements X, Y, Z of the Lie algebra $\mathfrak{g}$. Then $\xi(F) = K_*(X)F$, $\eta(F) = K_*(Y)F$, $\zeta(F) = K_*(Z)F$, and we obtain

$$\begin{aligned} \sigma(\eta, \zeta) &= \langle F, [Y, Z]\rangle, \\ \xi\sigma(\eta, \zeta) = \langle K_*(X)F, [Y, Z]\rangle &= -\langle F, [X, [Y, Z]]\rangle, \\ [\xi, \eta] &= -K_*([X, Y])F, \\ \sigma([\xi, \eta], \zeta) &= -\langle F, [[X, Y], Z]\rangle. \end{aligned}$$

Therefore, by the Jacobi identity

$$d\sigma(\xi, \eta, \zeta) = 2 \circlearrowleft \langle F, [X, [Y, Z]]\rangle = 0$$

Since G acts transitively on Ω, the vectors $K_*(X)F$, $X \in \mathfrak{g}$, span the whole tangent space $T_F\Omega$. Thus, $d\sigma = 0$.

□

5. Invariant polarization on homogeneous symplectic manifolds

In representation theory of Lie groups one is interested mainly in G-invariant polarizations of homogeneous symplectic G-manifolds. We know that the latter are essentially coadjoint orbits. Let G be a connected Lie group, and let $\Omega \subset \mathfrak{g}^*$ be a coadjoint orbit of G. Choose a point $F \in \Omega$.

Definition 5.1. *We say that a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is* ***subordinate*** *to a functional $F \in \mathfrak{g}^*$ if one of the following equivalent conditions are satisfied:*

(1) $F|_{[\mathfrak{h},\mathfrak{h}]} = 0$
(2) *the map $X \mapsto \langle F, X\rangle$ is a 1-dimensional representation of $\mathfrak{h}$.*

Note that the codimension of $\mathfrak{h}$ in $\mathfrak{g}$ is at least $\frac{1}{2} rkB_F$.

Definition 5.2. *We say that $\mathfrak{h}$ is a* ***real algebraic polarization*** *of F if in addition the condition $codim_{\mathfrak{g}}\mathfrak{h} = \frac{1}{2} rkB_F$ (i.e. $\mathfrak{h}$ has maximal possible dimension $\frac{dim\mathfrak{g}+rk\mathfrak{g}}{2}$)*

There is similarly the notion of complex algebraic polarization.

Definition 5.3. *An algebraic polarization $\mathfrak{h}$ is called* ***admissible*** *if it is invariant under the adjoint action of $Stab(F)$. Note that any polarization contains the Lie algebra $stab(F)$, hence is invariant under the adjoint action of $Stab^o(F)$, the connected component of identity in $Stab(F)$.*

It can happen that there is no real G-invariant polarization for a given $F \in \mathfrak{g}^*$. However, real G-invariant polarizations always exist for nilpotent and completely solvable Lie algebras. The relation between algebraic and geometric polarization is illustrated by the following theorem.

Lemma 5.4. *Let V be a real vector space endowed with a symplectic bilinear form B. Consider any filtration of V:*

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

where $dimV_k = k$. Denote by W_k the kernel of the restriction $B|_{V_k}$. Then

(1) *The subspace $W = \sum_k W_k$ is maximal isotropic for B.*
(2) *If in addition V is a Lie algebra, $B = B_F$ for some $F \in V^*$ and all V_k are ideals in V, then W is a polarization for F.*

Theorem 5.5. *There is a bijection between the set G-invariant real polarization P of a coadjoint orbit $\Omega \subset \mathfrak{g}^*$ and the set of admissible real algebraic polarization $\mathfrak{h}$ of a given element $F \in \Omega$. To a polarization $P \subset T\Omega$ there corresponds the algebraic polarization $\mathfrak{h} = (p_F)^{-1}(P(F))$.*

6. Orbit Method for Nilpotent Lie Groups

We will introduce some definitions and properties of nilpotent Lie groups and Lie algebras.

Definition 6.1. *A Lie algebra $\mathfrak{g}$ is called a* ***nilpotent Lie algebra*** *if it possesses the properties listed in the theorem below.*

Proposition 6.2. *The following properties of a Lie algebra $\mathfrak{g}$ are equivalent:*

(1) *There exists a sequence of ideals in $\mathfrak{g}$*

$$\{0\} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_{n-1} \subset \mathfrak{g}_n = \mathfrak{g}$$

such that

$$[\mathfrak{g}, \mathfrak{g}_k] \subset \mathfrak{g}_{k-1}, \quad 1 \leq k \leq n$$

(2) *The same as 1. with the additional property $dim \mathfrak{g}_k = k$.*
(3) *For any $X \in \mathfrak{g}$ the operator $ad\, X$ is nilpotent.*
(4) *$\mathfrak{g}$ has a matrix realization by strictly upper triangular matrices $X = ||X_{ij}||$, i.e. such that $X_{ij} = 0$ for $i \geq j$.*

Definition 6.3. *A connected Lie group G is called a* ***nilpotent Lie group*** *if its Lie algebra $\mathfrak{g}$ is nilpotent.*

Proposition 6.4. *The following properties of a connected Lie group are equivalent:*

(1) *G is a nilpotent Lie group.*
(2) *There exists a sequrence of connected normal subgroups*

$$\{e\} = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n = G$$

such that G_{k+1}/G_k is in the center of G/G_k.

(3) *Same as 2. with the additional property $dim\, G_k = k, \quad 1 \leq k \leq n$.*
(4) *G has a matrix realization by upper triangular matrices of the form $g = ||g_{ij}||$ satisfying*

$$g_{ij} = \begin{cases} 0 & \text{for } i = j \\ 0 & \text{for } i > j \end{cases}$$

The following feature of nilpotent Lie groups is important to us.

Theorem 6.5. *Let G be a connected and simply connected nilpotent Lie group. Then*

(1) *The exponential map $exp : \mathfrak{g} \to G$ is a diffeomorphism that establishes a bijection between subalgebras $\mathfrak{h} \subset \mathfrak{g}$ and closed connected subgroups $H \subset G$.*
(2) *In exponential cooridantes the gorup law is given by polynomial functions of degrees not exceeding the nilpotency class.*
(3) *G is unimodular and a two sided invariant measure dg is just the Lebesgue measure $d^n x = |dx_1 \wedge \cdots \wedge dx_n|$ in exponential coordinates.*
(4) *For any $F \in \mathfrak{g}^*$ we have $Stab(F) = expstab(F)$. Hence, $Stab(F)$ is connected and the coadjoint orbit $\Omega_f \cong G/Stab(F)$ is simply connected.*

The "Users Guide" rules can be proved for Nilpotent Lie groups using induction. The base case is when $dimG = 1$, i.e. $G \cong \mathbb{R}$. This can be easily verified.

Theorem 6.6 (Induction Theorem). *Assume that the rules are true for all connected and simply connected nilpotent Lie groups of dimension $< n$. Then they are valid also for a connected and simply connected nilpotent Lie group G of dimension n.*

Let Z be the center of G, and $\mathfrak{z}$ be the center of $\mathfrak{g}$. To prove that $\hat{G} = \mathcal{O}(G)$ as a set, it is enough

(1) to split the set $\hat{G}$ of all unirreps according to their restrictions to Z.
(2) to split the set $\mathcal{O}(G)$ of coadjoint orbits according to their projection to $\mathfrak{z}^*$.
(3) establish bijections between corresponding parts.

The fact that the Kirillov correspondence is a homeomorphism was proved by Brown [2]. In the rest of this section we will sketch a proof of this theorem:

Theorem 6.7 (Brown[2]). *For any connected, simply connected, nilpotent Lie group the Kirillov correspondence is a homeomorphism between $\hat{G}$ and $\mathcal{O}(G)$.*

Let $\tilde{G}$ denote the set of all equivalence classes of unitary (not necessarily irreducible) representations of a topological group G. The topology on $\tilde{G}$ is given below:

Definition 6.8. *A neighborhood of a representation (π, H) is determined by the following data:*

(1) *a compact subset $K \subset G$;*
(2) *a positive number ϵ;*
(3) *a finite collection of vectors $X = \{x_1, \cdots, x_n\} \in H$.*

The neighborhood $U_{K,\epsilon,X}(\pi)$ of (π, H) consists of all unitary represenations (ρ, V) such that there exists a finite family $Y = \{y_1, \cdots, y_n\}$ of vectors in V satisfying

$$|(\pi(g)x_i, x_j)_H - (\rho(g)y_i, y_j)_V| < \epsilon \quad \textit{for any } g \in K$$

The set $\hat{G}$ of unirreps is a subset of $\tilde{G}$ and inherits the topology from there. Also, recall the definition of weak containment of representations:

Definition 6.9. *A unirrep π is weakly contained in a unitary representation ρ if the point $\pi \in \hat{G} \subset \tilde{G}$ is contained in the closure of the point $\rho \in \tilde{G}$*

The problem of showing that the Kirillov correspondence is a homeomorphism csplits into two parts:

(1) prove that the map $\mathcal{O}(G) \to \hat{(G)}$ is continuous

(2) show that the inverse map is continuous

The first part is as follows. Suppose that a sequence of orbits $\{\Omega_n\}$ goes to a limit Ω. By definition of the quotient topology on $\mathcal{O}(G)$ this means that there exists a sequence of functionals $\{F_n\}$ such that $F_n \in \Omega_n$ and $lim_{n\to\infty} F_n = F \in \Omega$.

Let $\mathfrak{h}_n$ be a subalgebra of maximal dimension subordinated to F_n, and let $H_n = exp\mathfrak{h}_n$ be the corresponding subgroup of G. Passing to a subsequence if necessary, we can assume that all $\mathfrak{h}_n$ have the same codimension $2r$ and have a limit $\mathfrak{h}$ in the Grassmanian $G_{2r}(\mathfrak{g})$. It is clear that $\mathfrak{h}$ is subordiante to F. But it can happen that it does not have the maximal possible dimension (since rk B_F can be less that rk $B_{F_n} = 2r$)

Using Lemma 1.8 we can construct a subalgebra $\tilde{\mathfrak{h}}$ of maximal dimension subordiante to F so that $\mathfrak{h} \subset \tilde{\mathfrak{h}}$, hence $H = exp\mathfrak{h} \subset \tilde{H} = exp\tilde{\mathfrak{h}}$.

Denote by π (resp. $\tilde{\pi}$) the induced representation $Ind_H^G \rho_{F,H}$ (resp. the unirrep $\pi_\Omega = Ind_{\tilde{H}}^G \rho_{F,\tilde{H}}$). Here $\rho_{F,H}$ denotes the 1-dimension unirrep of H given by $\rho_{F,H}(expX) = e^{2\pi i \langle F,X\rangle}$ We need to show that $\tilde{\pi}$ is contained in the limit

$$lim_{n\to\infty}\pi_{\Omega_n} = lim_{n\to\infty} Ind_{H_n}^G \rho_{F_n,H_n}$$

Lemma 6.10. *We have $lim_{n\to\infty}\pi_{\Omega_n}$ in $\tilde{G}$ containes π_Ω.*

Sketch. Since the sequence $\mathfrak{h}_n$ tends to $\mathfrak{h}$ and all $\mathfrak{h}_n$ have the same dimension, we can identify $X_n = H_n \setminus G$ with the standard space $\mathbb{R}^n$, so that the actions of G on $\mathbb{R}^n$ arising from the identifications $\mathbb{R}^n \cong X_n$ have a limit. It is clear that this limit corresponds to the identification $\mathbb{R}^n \cong X = H \setminus G$. The lemma follows from Rule 2 of the User's Guide. □

In the case rk $B_F = 2r =$ rk B_{F_n} we have $\tilde{H} = H$, $\tilde{\pi} = \pi$ and we obtain the desired relation $lim_{n\to\infty}\pi_{\Omega_n} = \pi_\Omega$. Assume then that rk $B_F < 2r$, hence H is a proper subgroup in $\tilde{H}$. Then π is a reducible representation.

Lemma 6.11. *The represenation $\tilde{\pi}$ is weakly contained in π, hence is contained in $lim_{n\to\infty}\pi_{\Omega_n}$.*

Sketch. We observe that $\rho_{F,\tilde{H}}$ is weakly contained in $Ind_H^{\tilde{H}}\rho_{F,H}$. Then we use the following theorem proved by G.M.J. Fell.

Theorem 6.12 (Fell). *The induction functor Ind_H^G defines a continuous map from $\tilde{H}$ to $\tilde{G}$.*

□

Finally we look at the inverse map. We need the following concept:
$\mathfrak{g}_{n,k}$ is a universal nilpotent Lie algebra of nilpotency class k with n generators. By definiton, it is a quotient of a free Lie algebra with n generators by its k-th derivative spanned by all commutators of length $k+1$.

The approach by Brown is based on two simple observations:

(1) Any nilpotent Lie algebra can be considered as a quotient of $\mathfrak{g}_{n,k}$ for appropriate n and k. Consequently, the topological space $\hat{G}$ is a subspace of $\hat{G_{n,k}}$ with inherited topology.
(2) The Lie algebra $\mathfrak{g}_{n,k}$ has a huge group of automorphisms induced by automorphisms of the free Lie algebra.

The first observation reduces the general problem to the case $\mathfrak{g} = \mathfrak{g}_{n,k}$, and the second allows us to prove the theorem for $\mathfrak{g}_{n,k}$ by induction on k.

7. Example of Heisenberg Group

The heisenberg group H is defined as follows:

$$H = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \middle| \quad a, b, c \in \mathbb{R} \right\}$$

The Lie algebra is given by

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \middle| \quad x, y, z \in \mathbb{R} \right\}$$

Theorem 7.1 (Representation Theory methods). *$\hat{H}$ splits into two parts:*

(1) *1-parametric family of equivalence classes of infinite dimensional unirreps π_λ, $\lambda \neq 0$, where $\pi_\lambda(g_{a,b,c}) = e^{2\pi i \lambda c} v(\lambda b) u(\frac{a}{\hbar})$. Here $u(s)$ and $v(t)$ satisfy the CCR relation $u(s)v(t) = e^{ist\hbar} v(t)u(s)$. (Here $\hbar = \frac{h}{2\pi}$ where h is Planck constant)*
(2) *2-parametric family of 1-dimensional unirreps $\pi_{\mu,\nu}$ where $\pi_{\mu,\nu}(g_{a,b,c}) = e^{2\pi i(a\mu + b\nu)}$*

Now we calculate the coadjoint orbits of H. As mentioned in the introduction, $\mathfrak{h}^*$ can be identified with $M_3(\mathbb{R})/\mathfrak{h}^\perp$. We note that $\mathfrak{h}^\perp$ consisits of the upper triangular matrices. Hence $\mathfrak{h}^*$ can be identified with the strictly lower triangular matrices $F = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ z & y & 0 \end{pmatrix}$. Thus, the coadjoint action is given by

$$\begin{aligned} K(g)F &= p(gFg^{-1}) \\ &= \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ z & y & 0 \end{pmatrix} \begin{pmatrix} 1 & -a & -c + ab \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ x + bz & 0 & 0 \\ z & y - az & 0 \end{pmatrix} \end{aligned}$$

Hence, $K(g_{a,b,c})(x, y, z) = (x + bz, y - az, z)$. Thus we have the following:

Theorem 7.2. *The set $\mathcal{O}(H)$ of all coadjoint orbits for the Heisenberg group H consists of*

(1) *the 2-dimensional planes Ω_λ given by the equation $z = \lambda \neq 0$ and*
(2) *the points $\Omega_{\mu,\nu}$ given by the equations $x = \mu$, $y = \nu$, $z = 0$.*

Now we will compute the symplectic structure on the orbit Ω. The coadjoint action associates to the basic vectors X, Y, Z the vector fields $K_*(X), K_*(Y), K_*(Z)$ on $\mathfrak{h}^*$ which are tangent to Ω_λ:

$$K_*(X) = -z\partial_y, \quad K_*(Y) = z\partial_x, \quad K_*(Z) = 0$$

The symplectic form σ is defined by

$$\sigma(F)(K_*(X), K_*(Y)) = \langle F, [X, Y] \rangle \quad \text{or, } \sigma(x, y, z)(-z\partial_y, z\partial_x) = z$$

Therefore, we get

$$\sigma = \frac{1}{z} dx \wedge dy.$$

In conclusion, we have the following one-to-one correspondence between orbits and unirreps:

$$\Omega_\lambda \iff \pi_\lambda \qquad \Omega_{\mu,\nu} \iff \pi_{\mu,\nu}$$

References

[1] Kirillov, A. A. *Lectures on the orbit method*. Providence, RI: American Math. Soc., 2004.
[2] Brown, I.D. *Dual topology of a nilpotent Lie group*. Ann. Sci. Ecole Norm. Sup. 6 (1973), 407-411